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Harmonic maps and self-dual SU(3) gauge fields

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Abstract. The self-dual SU(3) Yang–Mills equations in Yang's R-gauge are expressed geometrically as a harmonic map of Riemannian manifolds. The study of geodesics and isometries in the manifold of fields provides certain new families of solutions of the Yang–Mills equations and a continuous group of transformations generating, from any family of solutions, a larger family with two additional parameters.

1. Introduction

Considerable progress has been made in recent years towards understanding self-dual Yang–Mills fields. In these studies two possible approaches have emerged. The first is geometrical: Ward (1977) and Atiyah and Ward (1977) succeeded in 'coding' the self-duality condition into the structure of certain algebraic vector bundles. Subsequently, Atiyah *et al* (1978) obtained a complete—but implicit—solution for all finite action self-dual Yang–Mills fields, leaving the open problem (apparently difficult) of finding an explicit parametrisation of the general solution. The second approach is analytic: Yang (1977) introduced a suitable gauge, the R-gauge, and reduced the self-dual SU(2) Yang–Mills equations to a system of three, relatively simple, equations which were found to be invariant under certain Bäcklund transformations (Corrigan *et al* 1978a, b, Morris 1980, Brihaye and Nuyts 1980). Prasad (1978), Ardalan (1978) and Brihaye *et al* (1978) clarified the algebraic meaning of the R-gauge and they generalised Yang's work to any semisimple gauge group. In particular, they have written explicitly the field equations for SU(3) gauge fields.

In this paper we pursue further the second approach by using the theory of harmonic maps. In a sense, we geometrise the second approach as well, since a central idea in the theory of harmonic maps is the construction of a suitable manifold with metric, the 'manifold of fields', in which we 'code' all the information of the field equations. The theory of harmonic maps was developed mainly by Eells and Samson (1964) and Eells and Lemaire (1978). In physics, it was used by Nutku (1974), Eris and Nutku (1975), Eris (1977) and Nutku and Halil (1977) in the search for solutions of the Einstein equation, by Misner (1978) in the consturction of various physical theories and by Nutku (1978) in the study of the SU(2) Yang–Mills fields. *Here we describe the SU(3) self-dual Yang–Mills fields as a harmonic map of Riemannian manifolds.* In addition, by the subsequent geometrical study of the manifold of fields, *we obtain certain new families of solutions of the Yang–Mills equations as well as a continuous group of*

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transformations which generate, from any family of solutions, an enlarged family with two additional parameters. We have found six independent Killing fields of the metric (8). The four 'obvious' Killing fields, by providing four integrals of motion for the geodesics, enable us to integrate the geodesic equations in the presence of the ansatz (16). The explicit knowledge of these geodesics of the metric (8) provided the solutions (27), (30), (31) and (32) of the self-dual equations. For the other two Killing fields, we have exponentiated the infinitesimal isometry they generate. The obtained (finite) isometry provided the transformations generating solutions from solutions.

2. The method

First we review the notion of harmonic maps.

Let (M, g_{mn}) and (N, g'_{AB}) be two C^∞ manifolds with metrics. We consider the mapping

$$f: M \to N, \qquad f = \{f^A\}, \tag{1}$$

where $\{f^A\}$ are coordinates on N. The 'energy functional' of the mapping (1) is given by (Eells and Lemaire 1978)

$$E(f) = \tfrac{1}{2} \int_M g'_{AB}\,(D_m f^A)(D_n f^B) g^{mn}\,\mathrm{d}\omega, \tag{2}$$

where D_m is the derivative operator and $\mathrm{d}\omega$ is a volume element on the manifold (M, g_{mn}). The mapping f is called harmonic if the corresponding energy functional is extremum. Writing the Euler–Lagrange equations for the functional (2), we conclude that the mapping $f: M \to N$ is harmonic if and only if

$$g^{mn} D_m D_n f^A + \Gamma^A_{BC}(D_m f^B)(D_n f^C) g^{mn} = 0, \tag{3}$$

where $A = 1, 2, \ldots, \dim N$, and Γ^A_{BC} are the Christoffel symbols of (N, g'_{AB}).

Mathematical physics usually leads to the study of systems of partial differential equations, and harmonic maps can be often used to provide a geometric description for these equations. Indeed, when the system of differential equations of the problem is of the form (3), one can choose (whenever it is possible) the manifolds (M, g_{mn}) and (N, g'_{AB}) and the mapping $f: M \to N$ such that the harmonicity conditions for f are precisely the differential equations of the problem. Thus, one codes into the geometric structures (M, g_{mn}) and (N, g'_{AB}) the details of the equations under consideration. (M, g_{mn}) is usually a space–time or flat Euclidean space; it will be referred to as the manifold of the independent variables. (N, g'_{AB}), whose local coordinates $\{f^A\}$ are precisely the unknown scalar fields of the differential system, will be referred as the manifold of fields. To describe a system of differential equations by a harmonic map, the difficult step is to find the metric g'_{AB} on the manifold of fields.

The study of the manifold of fields can provide a lot of information about the system. For instance, the knowledge of the geodesics of (N, g'_{AB}) provides particular solutions of the system while an isometry (Eisenhart 1926) on (N, g'_{AB}) leads to a method for generating additional solutions from solutions of the system (3). Indeed, it is easy to see that if $f^A = f^A(t)$ is an affinely parametrised geodesic of (N, g'_{AB}) and φ is a harmonic function on (M, g_{mn}) (i.e. a solution of $g^{mn} D_m D_n \varphi = 0$), then $\{f^A(\varphi)\}$ is a solution of the system (3). For the proof, notice that, for $f^A = f^A(\varphi)$, the left-hand sides of equations (3) become

$$D^m\left(\frac{\mathrm{d}f^A}{\mathrm{d}\varphi}D_m\varphi\right)+\Gamma^A_{BC}\frac{\mathrm{d}f^B}{\mathrm{d}\varphi}(D_m\varphi)\frac{\mathrm{d}f^C}{\mathrm{d}\varphi}(D^m\varphi)$$
$$=\left(\frac{\mathrm{d}^2f^A}{\mathrm{d}\varphi^2}+\Gamma^A_{BC}\frac{\mathrm{d}f^B}{\mathrm{d}\varphi}\frac{\mathrm{d}f^C}{\mathrm{d}\varphi}\right)(D^m\varphi)(D_m\varphi)+\frac{\mathrm{d}f^A}{\mathrm{d}\varphi}D^mD_m\varphi=0,$$

where the terms in parentheses vanish because $f^A(\varphi)$ is geodesic and the last term vanishes becauses φ is harmonic. it can also be seen that if $f^A \to \tilde{f}^A = \mathrm{i}(f^A)$ is an isometry on (N, g'_{AB}) then, for every solution $\{f^A\}$ of the system (3), $\{\tilde{f}^A\}$ is again a solution. For this proof we only have to observe that the isometry cannot possibly change the geodesics on (N, g'_{AB}) since it does not change the metric itself. The above two observations, combined with the fact that many systems of equations arising in physics can be described via harmonic maps, make harmonic maps a very useful tool in mathematical physics. We mention that the source-free Einstein (Matzner and Misner 1967) and Einstein–Maxwell equations with one or with two Killing fields, the SU(2) (Nutku 1978) and SU(3) self-dual Yang–Mills equations expressed in the R-gauge and the equations for the non-linear σ-models (Garber *et al* 1979) can be described in terms of suitable harmonic maps.

3. The SU(3) equations

The SU(3) self-dual Yang–Mills equations, written in the R-guage and in convenient notation, form the following system of eight coupled, second-order, nonlinear partial differential equations

$$D_a\bar{D}_a\varphi_1=\varphi_1^{-1}(D_a\varphi_1)(\bar{D}_a\varphi_1)-\varphi_2\varphi_1^{-1}(D_ax_1)(\bar{D}_ax_2)$$
$$-\varphi_2^{-1}(D_a\omega_1-y_1D_ax_1)(\bar{D}_a\omega_2-y_2\bar{D}_ax_2), \tag{4a}$$
$$D_a\bar{D}_a\varphi_2=\varphi_2^{-1}(D_a\varphi_2)(\bar{D}_a\varphi_2)-\varphi_1\varphi_2^{-1}(D_ay_1)(\bar{D}_ay_2)$$
$$-\varphi_1^{-1}(D_a\omega_1-y_1D_ax_1)(\bar{D}_a\omega_2-y_2\bar{D}_ax_2), \tag{4b}$$
$$D_a\bar{D}_ax_1=(D_ax_1)[\bar{D}_a\ln(\varphi_1^2/\varphi_2)]+\varphi_1\varphi_2^{-2}(D_a\omega_1-y_1D_ax_1)(\bar{D}_ay_2), \tag{4c}$$
$$D_a\bar{D}_ax_2=(\bar{D}_ax_2)[D_a\ln(\varphi_1^2/\varphi_2)]+\varphi_1\varphi_2^{-2}(\bar{D}_a\omega_2-y_2\bar{D}_ax_2)(D_ay_1), \tag{4d}$$
$$D_a\bar{D}_ay_1=(D_ay_1)[\bar{D}_a\ln(\varphi_2^2/\varphi_1)]-\varphi_2\varphi_1^{-2}(D_a\omega_1-y_1D_ax_1)(\bar{D}_ax_2), \tag{4e}$$
$$D_a\bar{D}_ay_2=(\bar{D}_ay_2)[D_a\ln(\varphi_2^2/\varphi_1)]-\varphi_2\varphi_1^{-2}(\bar{D}_a\omega_2-y_2\bar{D}_ax_2)(D_ax_1), \tag{4f}$$
$$D_a\bar{D}_a\omega_1=(D_ax_1)(\bar{D}_ay_1)+\varphi_1y_1\varphi_2^{-2}(D_a\omega_1-y_1D_ax_1)(\bar{D}_ay_2)$$
$$+(D_a\omega_1)[\bar{D}_a\ln(\varphi_1\varphi_2)]+y_1(D_ax_1)[\bar{D}_a\ln(\varphi_1/\varphi_2^2)], \tag{4g}$$
$$D_a\bar{D}_a\omega_2=(\bar{D}_ax_2)(D_ay_2)+\varphi_1y_2\varphi_2^{-2}(\bar{D}_a\omega_2-y_2\bar{D}_ax_2)(D_ay_1)$$
$$+(\bar{D}_a\omega_2)[D_a\ln(\varphi_1\varphi_2)]+y_2(\bar{D}_ax_2)[D_a\ln(\varphi_1/\varphi_2^2)]. \tag{4h}$$

In equations (4), φ_1, φ_2, x_1, x_2, y_1, y_2, ω_1 and ω_2 are scalar fields in four-dimensional Euclidean or Minkowskian space, depending on the interpretation of the corresponding gauge fields. The coordinates

$$
\begin{aligned}
y\sqrt{2} &= x_{(1)} + \mathrm{i}x_{(2)}, \qquad & \bar{y}\sqrt{2} &= x_{(1)} - \mathrm{i}x_{(2)}, \\
z\sqrt{2} &= x_{(3)} - \mathrm{i}x_{(4)}, \qquad & \bar{z}\sqrt{2} &= x_{(3)} + \mathrm{i}x_{(4)}
\end{aligned} \tag{5}
$$

have been used, where $x_{(\mu)}$ ($\mu = 1, 2, 3, 4$) are Cartesian coordinates. The derivative operators are

$$
D_a = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \qquad \bar{D}_a = \left(\frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \bar{z}}\right) \qquad (a = 1, 2), \tag{6}
$$

and repeated indices imply summation. The reality of the corresponding gauge fields requires that φ_1 and φ_2 are real while x_1, y_1 and ω_1 are complex conjugates of x_2, y_2 and ω_2 respectively. (Equations (4) are equivalent to equations (A5)–(A12) of Prasad (1978); to compare, set $x_1 = \rho_1$, $x_2 = \bar{\rho}_1$, $y_1 = \rho_2$, $y_2 = \bar{\rho}_2$, $\omega_1 = \rho_3$, $\omega_2 = \bar{\rho}_3$. The relation between the gauge fields and the present variables can also be found in Prasad (1978).)

Next we describe equations (4) as a harmonic map. First we observe that by assuming that $y = \bar{y}$ and $z = \bar{z}$, equations (4) are of the form (3), where g_{mn} is the flat metric with line element

$$
\mathrm{d}s^2 = g_{mn}(\mathrm{d}x^m)(\mathrm{d}x^n) = (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \tag{7}
$$

on the two-dimensional manifold $M = R^2$ (with coordinates (y, z)) and $\{f^A\}_{A=1,\ldots,8}$ stands for $(\varphi_1, \varphi_2, x_1, x_2, y_1, y_2, \omega_1, \omega_2)$. We will call the system with $y = \bar{y}$ and $z = \bar{z}$ the auxiliary system. Moreover, we have found† that the Christoffel symbols of the metric g'_{AB} with line element

$$
\begin{aligned}
\mathrm{d}s^2 &= g'_{AB}(\mathrm{d}f^A)(\mathrm{d}f^B) \\
&= \frac{1}{\varphi_1^2}(\mathrm{d}\varphi_1)^2 + \frac{1}{\varphi_2^2}(\mathrm{d}\varphi_2)^2 - \frac{1}{\varphi_1\varphi_2}(\mathrm{d}\varphi_1)(\mathrm{d}\varphi_2) + \left(\frac{y_1y_2}{\varphi_1\varphi_2} + \frac{\varphi_2}{\varphi_1^2}\right)(\mathrm{d}x_1)(\mathrm{d}x_2) \\
&\quad - \frac{y_1}{\varphi_1\varphi_2}(\mathrm{d}x_1)(\mathrm{d}\omega_2) - \frac{y_2}{\varphi_1\varphi_2}(\mathrm{d}\omega_1)(\mathrm{d}x_2) + \frac{\varphi_1}{\varphi_2^2}(\mathrm{d}y_1)(\mathrm{d}y_2) + \frac{1}{\varphi_1\varphi_2}(\mathrm{d}\omega_1)(\mathrm{d}\omega_2)
\end{aligned} \tag{8}
$$

are precisely the coefficients of the terms in the equations (4) which are quadratic in the gradients of the fields. We conclude, therefore, that the auxiliary system can be represented by the harmonic map $f: M \to N$, where the manifolds M and N are two- and eight-dimensional and their metrics are given by equations (7) and (8) respectively.

We now claim that both the applications of harmonic maps mentioned at the end of § 2 can be used in studies of the original system (4) from the mere knowledge of the description as a harmonic map of the auxiliary system. Indeed, when $f^A = f^A(t)$ is an affinely parametrised geodesic of the metric (8) and φ is a harmonic function in the *four-dimensional* Euclidean or Minkowskian space (i.e. a solution of $D_a\bar{D}_a\varphi = 0$), $f^A(\varphi)$ is obviously a solution of the self-dual equations (4). Moreover, any method for algebraically generating solutions from solutions for the auxiliary system obtained from

† It was not an easy task to find the metric g'_{AB}; we had to look for solutions of a system of 36 partial differential equations with 36 unknowns, the coefficients of the metric. Unfortunately the computations are too long to be included here. In the search for the solution our policy was to seek for the simplest non-trivial solution.

an isometry of the manifold (N, g'_{AB}) is also a method for generating solutions of the original system (4). In §§ 4 and 5 these two ideas are applied in the study of the self-dual equations (4).

4. Geodesics of g'_{AB} solutions

In this section we demonstrate the integration of the geodesic equations of the metric g'_{AB} with the ansatz (16).

It is immediate to write the geodesic equations of g'_{AB}. Indeed, since the metric g'_{AB} was found as the metric whose Christoffel symbols are precisely the coefficients of the quadratic in the gradients terms of the equations (4), to write the geodesic equations one simply has to consider the eight unknowns in (4) as functions of one variable (the affine parameter t) and interpret the gradients as first derivatives and the Laplacians as second derivatives. The knowledge of the metric g'_{AB}, however, provides additional information: since g'_{AB} admits the four 'obvious' Killing fields

$$x_1^a=\left(\frac{\partial}{\partial x_1}\right)^a,\qquad x_2^a=\left(\frac{\partial}{\partial x_2}\right)^a,\qquad \omega_1^a=\left(\frac{\partial}{\partial \omega_1}\right)^a,\qquad \omega_2^a=\left(\frac{\partial}{\partial \omega_2}\right)^a, \tag{9}$$

there are four integrals of motion of the geodesic equations. They are

$$\left(\frac{y_1y_2}{\varphi_1\varphi_2}+\frac{\varphi_2}{\varphi_1^2}\right)\dot{x}_1-\frac{y_2}{\varphi_1\varphi_2}\dot{\omega}_1=a_1,\qquad \left(\frac{y_1y_2}{\varphi_1\varphi_2}+\frac{\varphi_2}{\varphi_1^2}\right)\dot{x}_2-\frac{y_1}{\varphi_1\varphi_2}\dot{\omega}_2=a_2,$$
$$(\varphi_1\varphi_2)^{-1}(y_1\dot{x}_1-\dot{\omega}_1)=b_1,\qquad (\varphi_1\varphi_2)^{-1}(y_2\dot{x}_2-\dot{\omega}_2)=b_2, \tag{10}$$

where the dot denotes differentiation with respect to the affine parameter t. Using these integrals, we can eliminate $\dot{x}_1$, $\dot{x}_2$, $\dot{\omega}_1$ and $\dot{\omega}_2$ from the equations. The other four equations become

$$(\ln \dot{z}_1\varphi_1\varphi_2^{-2})^{\cdot}+bz_1\varphi_1\varphi_2(\dot{z}_1)^{-1}=0, \tag{11}$$
$$(\ln \dot{z}_2\varphi_1\varphi_2^{-2})^{\cdot}+bz_2\varphi_1\varphi_2(\dot{z}_2)^{-1}=0, \tag{12}$$
$$(\ln \varphi_1)^{\cdot\cdot}+\varphi_1^2\varphi_2^{-1}z_1z_2+b\varphi_1\varphi_2=0, \tag{13}$$
$$(\ln \varphi_2)^{\cdot\cdot}+b^{-1}\varphi_1\varphi_2^{-2}\dot{z}_1\dot{z}_2+b\varphi_1\varphi_2=0, \tag{14}$$

where we have set

$$z_1=a_2-b_2y_1,\qquad z_2=a_1-b_1y_2, \tag{15}$$

and $b=b_1b_2$.

We concentrate on the equations (11)–(14), whose integration is the only non-trivial step.

First, we look for solutions of equations (11)–(14) by making the ansatz

$$\varphi_1/\varphi_2=\mathrm{e}^{\alpha t+\beta}. \tag{16}$$

By subtracting equations (13) and (14) we obtain that

$$\dot{z}_1\dot{z}_2=bz_1z_2\varphi_1\varphi_2. \tag{17}$$

Then, by substituting $\varphi_1\varphi_2$ in the second terms of equations (11) and (12), we observe that these equations can be integrated once. They give

$$\dot{z}_1 z_2 \varphi_1 = k_1 \varphi_2^2, \tag{18}$$

$$z_1 \dot{z}_2 \varphi_1 = k_2 \varphi_2^2, \tag{19}$$

where k_1 and k_2 are two new constants.

Now by multiplying equations (18) and (19) and using equations (16) and (17), we obtain

$$z_1 z_2 = (k_1 k_2 b^{-1})^{1/2} \exp[-\tfrac{3}{2}(\alpha t + \beta)], \tag{20}$$

while by dividing equations (18) and (19) we obtain

$$z_1^{k_2} = \mathrm{e}^c z_2^{k_1}, \tag{21}$$

with c another constant. Equations (20) and (21) determine z_1 and z_2 (except when $k_2 = -k_1$):

$$z_1 = \exp\left\{-\frac{3k_1}{2(k_1+k_2)}(\alpha t + \beta) + \frac{1}{k_1+k_2}\left[c + \frac{k_1}{2}\ln\left(\frac{k_1 k_2}{b}\right)\right]\right\}, \tag{22}$$

$$z_2 = \exp\left\{-\frac{3k_2}{2(k_1+k_2)}(\alpha t + \beta) + \frac{1}{k_1+k_2}\left[-c + \frac{k_2}{2}\ln\left(\frac{k_1 k_2}{b}\right)\right]\right\}. \tag{23}$$

Then it is straightforward to determine φ_1 and φ_2 by using equations (16) and (17). We obtain

$$\varphi_1 = \frac{3\alpha\varepsilon}{2(k_1+k_2)}\left(\frac{k_1 k_2}{b}\right)^{1/2} \exp[\tfrac{1}{2}(\alpha t + \beta)], \tag{24}$$

$$\varphi_2 = \frac{3\alpha\varepsilon}{2(k_1+k_2)}\left(\frac{k_1 k_2}{b}\right)^{1/2} \exp[-\tfrac{1}{2}(\alpha t + \beta)], \tag{25}$$

where $\varepsilon = \pm 1$. The expressions (22)–(25) satisfy the equations (13) and (14) provided that the constants are subject to

$$2(k_1 + k_2) + 3\varepsilon a b = 0. \tag{26}$$

Finally, by using equations (10) one easily obtains x_1, x_2, ω_1 and ω_2. By setting now $\chi = \exp[\frac{1}{2}(\alpha t + \beta)]$, redefining the parameters and restricting them to satisfy the reality condition, we obtain the following family of solutions to system (4):

$$\begin{aligned}
\varphi_1 &= -(mm^*)^{2/3}(qq^*bb^*)^{-1/3}\chi, & \varphi_2 &= -(mm^*qq^*)^{1/3}(bb^*)^{-2/3}\chi^{-1},\\
x_1 &= mq^{-1}\chi^{m^*}, & x_2 &= (mq^{-1})^*\chi^{m},\\
y_1 &= b^{-1}(a - q\chi^{-m^*}), & y_2 &= (b^*)^{-1}(a^* - q^*\chi^{-m}),\\
\omega_1 &= amq^{-1}b^{-1}\chi^{m^*}, & \omega_2 &= (amq^{-1}b^{-1})^*\chi^{m},
\end{aligned} \tag{27}$$

where $D_a\bar{D}_a\,(\ln\chi) = 0$, a, b, q, m are complex constants with $m + m^* = 3$ and the asterisk denotes complex conjugation.

Next we examine the case $k_2 = -k_1 = k$ (say) for which equation (21) implies that $z_1 z_2 = \mathrm{e}^{c/k}$ which, compared with equation (20), gives that $\alpha = 0$ and $\exp(2ck^{-1} + 3\beta) = -k^2 b^{-1}$. In this case, in addition, we have $\varphi_1/\varphi_2 = \mathrm{e}^{\beta}$. Then it is easy to see that equations (17), (18) and (19) reduce to

$$(\ln z_1)^{\cdot} = -k\varphi_2 \exp(-\beta - ck^{-1}), \tag{28}$$

while equations (13) and (14) reduce to

$$(\ln \varphi)^{\cdot\cdot} + kp^{-1}\varphi - p^2\varphi^2 = 0, \tag{29}$$

where $b = -p^2$ and $\varphi = \varphi_1 \exp(-\beta/2) = \varphi_2 \exp(\beta/2)$. Equation (29) admits the solutions

$$\varphi = 2v^2\,e^{-vt}[(e^{-vt} + kp^{-1})^2 - p^2v^2]^{-1}$$

and

$$\varphi = -pv^2[k + (k^2 + p^4v^2)^{1/2}\sin vt]^{-1},$$

where v is a constant. (We have ignored the second constant of integration, which can be introduced by the change $vt \to vt + \varepsilon$. Also, we have ignored the inclusion of a singular solution of equation (29) which does not lead to any additional solution of the system (4).) Then it is easy to complete the solution for the geodesics. From these we obtain, with the substitutions $e^{-vt} + kp^{-1} \to \chi$ and $\tan(vt/2) \to \chi$, respectively, and redefinitions of various parameters, the following two families of solutions of the system (4):

$$\begin{aligned}
&\frac{\varphi_1}{c^2} = \varphi_2 c^2 = \frac{2v^2(\chi - kk^*)}{\chi^2 + p^2v^2}, \qquad x_1 = \frac{2vc^3k^*}{\chi + ipv}, \qquad x_2 = \frac{2vc^3k}{x - ipv},\\
&y_1 = \left(a^* - k\frac{\chi + ipv}{\chi - ipv}\right)(c^3b^*)^{-1}, \qquad y_2 = \left(a - k^*\frac{\chi - ipv}{\chi + ipv}\right)(c^3b)^{-1},\\
&\omega_1 = v(b^*)^{-1}\left(\frac{2a^*k^* - kk^* - ipv}{\chi + ipv} + \frac{ipv - kk^*}{\chi - ipv}\right),\\
&\omega_2 = vb^{-1}\left(\frac{2ak - kk^* + ipv}{\chi - ipv} - \frac{kk^* + ipv}{\chi + ipv}\right),
\end{aligned} \tag{30}$$

where v, c and p are real and k, a, b are complex constants with $bb^* = p^2$ and $D_a\bar{D}_a \ln(\chi - kk^*) = 0$, and

$$\begin{aligned}
&\frac{\varphi_1}{c^2} = \varphi_2 c^2 = \frac{kk^*(a^2-1)^2(\chi^2+1)}{4a^2p^2(\chi + a)(\chi + a^{-1})},\\
&x_1 = -ik^*c^3(a^2-1)(ap)^{-1}(\chi + a)^{-1}, \qquad x_2 = -ikc^3(a^2-1)(ap)^{-1}(\chi + a^{-1})^{-1},\\
&y_1 = \left(\alpha^* - k\frac{\chi + a}{\chi + a^{-1}}\right)(c^3b^*)^{-1}, \qquad y_2 = \left(\alpha - k^*\frac{\chi + a^{-1}}{\chi + a}\right)(c^3b)^{-1},\\
&\omega_1 = \frac{ik^*(a^2-1)}{4apb^*}\left(\frac{(a^2+1)k - 4\alpha^*}{\chi + a} + \frac{k(a^2+1)}{a^2(\chi + a^{-1})}\right),\\
&\omega_2 = \frac{ik(a^2-1)}{4apb}\left(\frac{(a^2+1)k^* - 4a^2\alpha^*}{a^2(\chi + a^{-1})} + \frac{k^*(a^2+1)}{\chi + a}\right),
\end{aligned} \tag{31}$$

where $D_a\bar{D}_a(\tan^{-1}\chi) = 0$, p, c are real and a, α, b, k are complex constants subject to $aa^* = 1$ and $bb^* = p^2$.

Finally, for $b_1 = b_2 = 0$ (equation (10)) we were able to integrate the geodesic equations only for $\varphi_1 = \varphi_2 = \varphi$. This case leads to $(\ln \varphi)^{\cdot\cdot} + a_1a_2\varphi = 0$ which, when integrated, provides the last two families of solutions:

$$\begin{aligned}
&\varphi_1=\varphi_2=v^2(2aa^*)^{-1}(1-\chi^2), \qquad x_1=v(a^*)^{-1}\chi, \qquad x_2=va^{-1}\chi,\\
&y_1=v(c^*)^{-1}\chi+\beta^*, \qquad y_2=vc^{-1}\chi+\beta,\\
&\omega_1=v^2(2a^*c^*)^{-1}(\chi^2-1)+\beta^*v(a^*)^{-1}\chi,\\
&\omega_2=v^2(2ac)^{-1}(\chi^2-1)+\beta va^{-1}\chi,
\end{aligned} \tag{32}$$

where $D_a\bar{D}_a\,[\ln(\chi-1)/(\chi+1)]=0$, v is a real and a, β, c are complex constants subject to $aa^*=cc^*$, and the family obtained from equations (32) by the formal substitution $v\to -\mathrm{i}v$, $\chi\to \mathrm{i}\chi$, where now χ satisfies $D_a\bar{D}_a\,(\tan^{-1}\chi)=0$.

In writing the solutions (27), (30), (31) and (32) we have systematically ignored the inclusion of additional constants to x_1, x_2, ω_1 and ω_2 as well as the constants which could have been introduced by considering the general solution of the scalars which satisfy Laplace's equation. The solutions (27), (30), (31) and (32) are missing among the explicitly known solutions we are aware of.

5. Isometries of g'_{AB} transformations

In this section we use two Killing fields ξ^a and ζ^a of the metric (8), in addition to the four 'obvious' Killing fields, and obtain a method for generating, from any solution of the system (4), a new solution with two additional free parameters. The four 'obvious' Killing fields which, by providing the integrals (10), helped in the integration of the geodesic equations lead only to trivial transformations, namely, the transformations describing the addition of constants to the four fields x_i and ω_i.

In the coordinate system $\{\varphi_i, x_i, y_i, \omega_i, i=1,2\}$ the contravariant components for these two Killing fields are

$$\begin{aligned}
\xi^a&=(0,-\varphi_2y_1,\omega_1,0,-y_1^2,\varphi_2^2\varphi_1^{-1},0,0),\\
\zeta^a&=(0,-\varphi_2y_2,0,\omega_2,\varphi_2^2\varphi_1^{-1},-y_2^2,0,0).
\end{aligned} \tag{33}$$

We consider the isometry infinitesimally described by the linear combination $\alpha_1\xi^a+\alpha_2\zeta^a$, where the α_i's are two arbitrary constants. The crucial step, of course, is the exponentiation of this infinitesimal isometry to a finite one, i.e. a step which requires the integration of the system of the ordinary differential equations

$$\begin{aligned}
&\dot\varphi_1=0, &&\dot\varphi_2=-\varphi_2(\alpha_1y_1+\alpha_2y_2),\\
&\dot x_1=\alpha_1\omega_1, &&\dot x_2=\alpha_2\omega_2,\\
&\dot y_1=-\alpha_1y_1^2+\alpha_2\varphi_2^2\varphi_1^{-1}, &&\dot y_2=-\alpha_2y_2^2+\alpha_1\varphi_2^2\varphi_1^{-1},\\
&\dot\omega_1=0, &&\dot\omega_2=0.
\end{aligned} \tag{34}$$

In this section the dot denotes differentiation with respect to a parameter t along the isometry. Only the integration of the subsystem of the second, fifth and sixth of the equations (34) is non-trivial and it is performed as follows. We first observe that the equations admit the integral $\varphi_2(\alpha_1y_1-\alpha_2y_2)^{-1}=b$, for some constant b. Next, by multiplying the last two equations by α_1 and α_2 respectively, adding them, eliminating φ_2 and expressing the y_i's in terms of the new variable z via $\alpha_1y_1-\alpha_2y_2=z^{-1}$ and $\alpha_1y_1+\alpha_2y_2=\dot z z^{-1}$, we obtain the equation $2z\ddot z-\dot z^2=4k-1$, where k is the constant

$k=\alpha_1\alpha_2 b^2\varphi_1^{-1}$; this equation admits the integral $(\dot{z}^2+4k-1)z^{-1}=$ constant and it can be integrated easily. Finally, by expressing the integration constants in terms of the $t=0$ values of the fields and considering the $t=1$ image point of the isometry, we obtain the transformation generating from solutions other solutions with the two additional parameters α_1 and α_2:

$$\begin{aligned}
&\tilde{\varphi}_1=\varphi_1, \qquad \tilde{\varphi}_2=\Omega^{-1}\varphi_2,\\
&\tilde{x}_1=x_1+\alpha_1\omega_1, \qquad \tilde{x}_2=x_2+\alpha_2\omega_2,\\
&\tilde{y}_1=\Omega^{-1}[y_1+\alpha_2(y_1y_2+\varphi_2^2\varphi_1^{-1})],\\
&\tilde{y}_2=\Omega^{-1}[y_2+\alpha_1(y_1y_2+\varphi_2^2\varphi_1^{-1})],\\
&\tilde{\omega}_1=\omega_1, \qquad \tilde{\omega}_2=\omega_2,
\end{aligned} \tag{35}$$

where

$$\Omega=1+\alpha_1y_1+\alpha_2y_2+\alpha_1\alpha_2(y_1y_2+\varphi_2^2\varphi_1^{-1}).$$

It is straightforward to verify that equations (35) generate a two-dimensional group of transformations. These transformations preserve the combination $(\alpha_1y_1-\alpha_2y_2)\varphi_2^{-1}$. For any choice of complex conjugate constants α_1 and α_2 the transformations also preserve the reality of the corresponding Yang–Mills fields.

We could combine the transformations (35) with the addition of constants to the x_i's and ω_i's. To obtain some feeling about the resulting transformations, we study the Lie algebra structure of the corresponding Killing fields: $\{\xi^a, \zeta^a, x_1^a, x_2^a, \omega_1^a, \omega_2^a\}$ from a Lie algebra, $\{\xi^a, \zeta^a, x_1^a, x_2^a\}$ is a commutative subalgebra (in fact, an ideal), while $\{\xi^a, \zeta^a, \omega_1^a, \omega_2^a\}$ is not commutative and it is not a subalgebra. Successive applications, therefore, of the transformations (35) and the additions of constants to ω_i's could in fact provide the entire six-parameter group of transformations which we expect to construct from the knowledge of the six independent Killing fields.

6. Discussion

The information about a system of differential equations is coded in the metric g'_{AB} of the manifold of fields in the same way as it can be coded in a Lagrangian in the more conventional approaches of theoretical physics. In fact there exists a close relationship (R Geroch 1980, private communication) between the metric g'_{AB} and the Lagrangian L which describe the same system of differential equations: Let $f:M\to N$ be a C^∞ mapping of manifolds with metrics g_{ab} and g'_{AB} respectively. Consider the pull back (Abraham and Marsden 1978) $(f_* g'_{AB})_{ab}$ of g'_{AB}, which is a tensor field on M, and take its trace $L=g^{ab}(f_* g'_{AB})_{ab}$. The Euler–Lagrange equations on the scalar L are precisely the harmonicity conditions (3) on the mapping f (R Geroch 1980, private communication). When $\{x^a\}$ and $\{f^A\}$ are local coordinate systems in M and N respectively, we can obtain for the Lagrangian the expression

$$L=g^{ab}g'_{AB}(\partial f^A/\partial x^a)(\partial f^B/\partial x^b). \tag{36}$$

Because g^{ab} is usually known, all the esssential information about the metric g'_{AB} is contained in the Lagrangian L, and vice versa. In fact, had we known it, it would have been possible to obtain the metric g'_{AB} from the Lagrangian for the self-dual Yang–Mills equations obtained by Brihaye *et al* (1978). However, there is an advantage in thinking in terms of harmonic maps, i.e. the metric g'_{AB} on the manifold of fields, instead of Lagrangians—the scalar (36) on the manifold of the independent variables. By discarding the terms g^{ab}, $\partial f^A/\partial x^a$ and $\partial f^B/\partial x^b$ from the scalar (36), we focus our attention on the metric g'_{AB} which involves the only essential information. In addition, having isolated the geometrical object (N, g'_{AB}), we can ask certain geometrical questions (for instance, about geodesics, Killing fields, finite isometries) which provide a great deal of information about the original system of differential equations.

The families of solutions obtained in § 4 can be characterised by the fact that they are functionally dependent; more generally, any solution obtained by using harmonic maps and the geodesics on the manifold of fields is functionally dependent. It should be clear, therefore, that the families (27), (30), (31) and (32) present only a small portion of the solutions admitted by the system (4). In fact, having made the ansatz (16) for the integration of the geodesic equations, we have not even obtained all the functionally dependent solutions.

In the present search for solutions of the system (4), all our considerations were local; it seems a formidable task to try to incorporate in the search for the geodesics on the manifold of fields the requirements for the correct asymptotic behaviour and for the absence of singularities in the resulting Yang–Mills fields. By looking, however, in the explicit expressions of the obtained solutions, we can see a necessary condition for the action of the corresponding Yang–Mills fields to be finite: the four scalar fields mentioned immediately after equations (27), (30), (31) and (32) which satisfy the Laplace equation should behave asymptotically like $O(r^{-1})$. Therefore, from our experience with the Laplace equation we expect that these solutions will be singular exactly at the origin $r = 0$.

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